

On infrared problems of effective Lagrangians of massive spin 2 fields coupled to gauge fields

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Abstract

In this paper we analyze the interactions of a massive spin-2 particles charged under both Abelian and non-Abelian group using the Porrati-Rahman Lagrangian. This theory is valid up to an intrinsic cutoff scale. Phenomenologically a theory valid up to a cutoff scale is sensible as all known higher spin particles are non-fundamental and it is shown that indeed this action can be used to estimate some relevant cross section. Such action necessarily includes Stückelberg field and therefore it is necessary to fix the corresponding gauge symmetry. We show that this theory, when the Stückelberg symmetry is gauge-fixed, possesses a non-trivial infrared problem. A gauge fixing ambiguity arises which is akin to the Gribov problem in QCD in the Abelian case as well. In some cases (such as when the space-time is the four-dimensional torus) the vacuum copies can be found analytically. A similar phenomenon also appears in the case of Proca fields. A very interesting feature of these copies is that they arise only for "large enough" gauge potentials. This opens the possibility to avoid the appearance of such gauge fixing ambiguities by using a Gribov-Zwanziger like approach.

1 Introduction

The Standard Model has proven to be much more successful than originally expected. The 125 GeV boson recently observed by ATLAS and CMS at the LHC looks very much like the long time awaited Higgs boson. Nevertheless, this amazing success is today one of the most intriguing puzzles in particle physics. The resolution of the well known hierarchy and naturalness problems requires the existence of New Physics at a scale of a few TeV where new particles must appear. In general, signals for new spin 0, 1/2 and 1 states have been extensively studied. Nevertheless, particles with higher spins may also appear. In particular, new massive spin-2 particles are of phenomenological interest. A well known example is the Kaluza-Klein excitation of the graviton, predicted in models with extra dimensions. Less attention has been put on spin-2 particles that can appear as composite states formed by a pair of color-octet spin-1 fields (colorons) predicted in models like Top-Color, non-minimal Technicolor and Universal Extra Dimensions ([1]). Interestingly, in this last case, the massive spin-2 states may be colored. Additionally, in a more standard sector, one of the most interesting features of strong interactions and QCD is the existence of many massive higher spin resonances (such as like $\pi_2(1670)$, $\rho_3(1690)$ or $a_4(2040)$) which have a very important phenomenological role.

Local gauge invariance, which is one of the basic ingredients of the standard model gives a natural way to couple matter fields to a gauge field. For instance, in Quantum Electrodynamics, the electron

is coupled to the photon by replacing the partial derivative in the Dirac equation by a gauge covariant derivative. Naively one may expect that the rule of minimal coupling holds also for any higher spin field. However it has already been noticed long time ago by Fierz and Pauli [2] that this is not true. Indeed, by naively replacing the derivative with the covariant derivative in the equations of motion of any field with spin higher than one immediately gets an algebraic inconsistency with the equations of motion. To avoid such inconsistencies several attempts have been done to derive the equations of motion and the subsidiary conditions for arbitrary spin fields coupled to a gauge field from an action principle. The only way known up to now to perform this is by introducing auxiliary fields in the Lagrangian [2], [3],[4]. An explicit Lagrangian for the generic Bosonic case was proposed in [5].

Unfortunately, also the introduction of auxiliary fields in the Lagrangian does not leave the theory free of other severe pathologies [6], [7], [8]. One well known pathology is that if one insists on the minimal coupling the theory does not propagate the correct number of degrees of freedom.

One may be tempted to cure this pathology by introducing non-minimal couplings. Indeed there have been proposed phenomenological models for spin two field with spin-stress-energy tensor coupling [9] [10] [11] [12]. Another non-minimal coupling used in phenomenological models is a coupling quadratic in the spin field and linear in the field strength tensor [13]. Such a coupling is known as the Federbush model [14]. The introduction of such non-minimal couplings however introduces a new pathology known as the Velo-Zwanziger problem i.e. super-luminal propagation of the fields and therefore acausal behavior [7], [8].

On the theoretical side, higher spin particles (whose masses are of the order of M_{string}) have a fundamental role in string theory. In the context of string theory an interesting possibility to avoid the Velo Zwanziger problem, in a constant electromagnetic background, has been outlined in [16] [17]. On phenomenological side, all the experimentally observed higher spin particles are resonances rather than fundamental particles: consequently any local Lagrangian describing them is only valid up to some finite UV cutoff.

Porrati and Rahman [18] have shown that the Velo-Zwanziger problem can be associated to the existence of an intrinsic cutoff scale in the Lagrangian describing an interacting massive spin two particle. The authors analyzed the nature of the UV cutoff and showed that it is possible to construct an intrinsic, model independent UV cutoff. Their results are based on the use of the Stückelberg formalism since the Stückelberg fields allow to construct gauge-invariant interactions for charge massive spin-2 fields. It is important to stress that the Stückelberg formalism does not cure by itself the usual pathologies of interacting higher spin Lagrangians as they are related to the existence of an intrinsic cutoff. In order to cure these pathologies it is necessary to introduce non-minimal couplings and additional degrees of freedom. This has been done for example in [17] [19] for Lagrangians derived from String Theory. In [18] the cutoff scale can be pushed to a higher value by adding a new non-minimal coupling in form of a dipole term.

However in this paper we will show that the use of the Stückelberg formalism, introduces also a new problem associated to the gauge fixing. In principle, one could always gauge-fix the Stückelberg fields to zero using the Stückelberg gauge symmetry. However, one of the key technical points [18] is that, if one chooses *a suitable covariant gauge fixing*, it becomes possible to diagonalize the kinetic terms and to single out the sub-sector of the theory which is the source of all pathologies mentioned above. In other words, this unified description of phenomena such as strong coupling at finite energy scale, acausal propagation in external fields and so on within the Stückelberg formalism strictly relies

on a specific covariant gauge-fixing¹.

One of the goals of this paper is to show that this apparently harmless procedure to choose a covariant gauge-fixing actually hides a non-trivial IR problem. We will focus on the spin-2 case but our analysis suggests that similar results hold for spin higher than two. The free part of the Lagrangians describing the effective theory of a spin-2 charged massive particle are well known (see, for instance, [5] [21] [22] [23]). The coupling with gauge fields can be introduced as in [16] and [18]. The non-trivial IR problem which will be discussed here is related to the pathologies arising when the Stückelberg symmetry is gauge-fixed. It will be shown that the usual gauge fixing of the Stückelberg symmetry is ambiguous. Such pathologies correspond to gauge-fixing ambiguities of Gribov type [30] and affect the IR region of the theory. In the present manuscript, we will mainly analyze the ambiguities arising in the Abelian case (see, for instance, [31] and references therein). Interestingly enough, the way in which such Gribov-like "disease" appears in the present case suggests a very natural therapy as well.

The paper is organized as follows. First it is shown how the Porrati Rahman action can be used to estimate some phenomenologically relevant process involving non-fundamental massive spin particles. Then a basic review of gauge fixing problems in non-abelian gauge theory will be given. In the fourth section the gauge fixing problem specific to the Porrati Rahman action and the presence of Gribov copies will be discussed. The last section will be dedicated to the conclusions.

2 Effective Lagrangian for charged spin-2 particles

The standard Pauli-Fierz Lagrangian L_{PF} for a spin-2 massive field on flat space-times reads:

$$L_{PF} = -\frac{1}{2} (\partial_\mu h_{\nu\rho})^2 + (\partial_\mu h^{\mu\nu})^2 + \frac{1}{2} (\partial_\mu h)^2 - (\partial_\mu h^{\mu\nu}) (\partial_\nu h) - \frac{m^2}{2} (h_{\mu\nu}^2 - h^2) , \quad (1)$$

$$h = h^\mu{}_\mu . \quad (2)$$

The Stückelberg procedure corresponds, as explained in [18], to the replacement

$$h_{\mu\nu} \rightarrow \hat{h}_{\mu\nu} = h_{\mu\nu} + \frac{1}{m} \left[\partial_\mu \left(B_\nu - \frac{1}{2m} \partial_\nu \phi \right) + \partial_\nu \left(B_\mu - \frac{1}{2m} \partial_\mu \phi \right) \right] , \quad (3)$$

where B_ν and ϕ are the so-called Stückelberg fields accounting for the spin-1 and spin-0 degrees of freedom avoiding the well known singularities of [24]. With the above replacement, the Lagrangian becomes invariant under the following Stückelberg symmetry

$$\delta h_{\mu\nu} = \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu , \quad (4)$$

$$\delta B_\mu = \partial_\mu \lambda - m \lambda_\mu , \quad (5)$$

$$\delta \phi = 2m \lambda , \quad (6)$$

where λ and λ_μ are "gauge" parameters.

It is worth to point out that it is also possible to introduce a more generic Lagrangian as done in [25] where in Eq. (3) on the right hand side there is also a term proportional to $\eta_{\mu\nu} \phi$. This would add in the gauge transformation (4) also a term proportional to $\eta_{\mu\nu} \phi$. We will use however the Lagrangian of [18] due to its physical interest as it is obtained from a Kaluza-Klein compactification of the $d + 1$

¹The Stückelberg formalism also naturally arises when one analyzes the coupling of spin-2 excitations to gauge field via a Kaluza-Klein reduction [20]: in this context the so-called Stückelberg symmetry corresponds to the usual symmetry of a spin-2 particle in 5 dimensions when the theory is reduced to 4 dimensions.

dimensional the free theory. Moreover this choice does not imply any loss of generality in our discussion as it is straightforward to see that using the Lagrangian proposed in ([25]) would not change the results found in this paper (see section 4).

As it is by now well known (see for instance, [18] [16]), the use of the Stückelberg formalism appears to be unavoidable if one wants to include the interactions of massive spin-2 particles with gauge fields which is obviously important for phenomenological studies. Let us recall that color-octet spin-2 fields are expected to appear as composite states in many models like Top-color, Technicolor and Universal Extra Dimensions. Their production at the LHC was studied in [1] using general properties of bound states. Here, nevertheless, we want to start by constructing the gauge theory for a color-octet spin-2 massive particle and examine its properties.

Thus, following [18], let us consider spin-2 massive particles charged under a non-Abelian gauge group $h_{\mu\nu}^a$ (where a is the index corresponding to the Lie algebra of the gauge group, we will focus on $U(1)$ and $SU(N)$ whose structure constants will be denoted as f^{abc}). In this case also the Stückelberg fields B_μ^a and ϕ^a have to belong to the same representation of $h_{\mu\nu}^a$. Therefore, the Stückelberg symmetry becomes

$$\delta h_{\mu\nu}^a = (D_\mu \lambda_\nu)^a + (D_\nu \lambda_\mu)^a , \quad (7)$$

$$\delta B_\mu^a = (D_\mu \lambda)^a - m \lambda_\mu^a , \quad (8)$$

$$\delta \phi^a = 2m \lambda^a , \quad (9)$$

$$D_\mu = \partial_\mu + ie [A_\mu,] , \quad (10)$$

where A_μ is the gauge field to which we want to couple the massive spin-2 particle.

It is worth emphasizing here two very important differences which distinguish the gauge parameters of Stückelberg symmetry from the usual gauge parameters appearing in Maxwell and Yang-Mills theories. Firstly, the Stückelberg gauge parameter λ_μ carry a Lorentz index (unlike what happens in Maxwell and Yang-Mills theories). Secondly, the Stückelberg gauge parameters λ and λ_μ are also charged² under the $U(1)$ or $SU(N)$ gauge fields to which $h_{\mu\nu}$, B_μ and ϕ couples. This can be easily seen from Eqs. (7), (8) and (9). In particular, λ_μ^a has the same structure of B_μ^a and, likewise, λ^a of ϕ^a . This implies that, in the Abelian case, the Stückelberg gauge parameters λ_μ and λ are charged under the $U(1)$ gauge group (due to the fact that $h_{\mu\nu}$, B_μ and ϕ will couple to the Maxwell gauge potential A_μ). Consequently, in the Abelian case, the Stückelberg gauge parameters λ_μ and λ are complex. As it will be shown in the following sections, it is because of these two differences that, in the case of the Stückelberg gauge transformation, gauge fixing ambiguities appear in the Abelian case as well.

The replacement corresponding to Eq. (3) and the minimal coupled Lagrangian read

$$h_{\mu\nu}^a \rightarrow \hat{h}_{\mu\nu}^a = h_{\mu\nu}^a + \frac{1}{m} \left[\left(D_\mu \left(B_\nu - \frac{1}{2m} D_\nu \phi \right) \right)^a + \left(D_\nu \left(B_\mu - \frac{1}{2m} D_\mu \phi \right) \right)^a \right] , \quad (11)$$

$$L_{PF} = \text{tr} \left[-\frac{1}{2} \left((D_\mu \hat{h}_{\nu\rho})^a \right)^2 + \left((D^\mu \hat{h}_{\mu\nu})^a \right)^2 + \frac{1}{2} \left((D^\mu \hat{h})^a \right)^2 + \right. \\ \left. - \left(D^\mu \hat{h}_{\mu\nu} \right)^a \left(D^\nu \hat{h} \right)^a - \frac{m^2}{2} \left((\hat{h}_{\mu\nu}^a)^2 - (\hat{h}^a)^2 \right) \right] - \frac{\text{Tr}}{4} F_{\mu\nu} F^{\mu\nu} , \quad (12)$$

where $F_{\mu\nu} F^{\mu\nu}$ is the kinetic term for the gauge field A_μ . Hence, the Lagrangian in Eq. (12) describe the effective interactions of the massive spin-2 particle with a non-Abelian gauge field. As it has been

²On the other hand, in the Yang-Mills case, the gauge parameter only carries indices of the $SU(N)$ group itself while, in the Maxwell case, the gauge parameter is a real scalar function (and so it carries no $U(1)$ charge at all).

shown in [18] the gauge-fixing terms here below is the best one to study the spectrum of the free theory:

$$L_{gf1} = -2 \left(\partial^\nu h_{\mu\nu}^a - \frac{1}{2} \partial_\mu h^a + m B_\mu^a \right) \left(\partial^\lambda h_{\sigma\lambda}^a - \frac{1}{2} \partial_\sigma h^a + m B_\sigma^a \right) g^{\mu\sigma} , \quad (13)$$

$$L_{gf2} = -2 \left(\partial^\nu B_\nu^a + \frac{m}{2} (h^a - 3\phi^a) \right)^2 , \quad (14)$$

then the kinetic terms L_{free} get canonical forms

$$L_{free} = h_{\mu\nu}^a (\square - m^2) h^{a\mu\nu} - \frac{1}{2} h^a (\square - m^2) h^a + 2 B_\mu^a (\square - m^2) B^{a\mu} + \frac{3}{2} \phi^a (\square - m^2) \phi^a .$$

To describe the interacting case, one needs to replace derivatives with covariant derivatives. Hence, the total Lagrangian can be written as

$$L = L_{free} - \frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} + L_{int} , \quad (15)$$

where L_{int} encodes the interactions terms which can be found by simply expanding explicitly the expressions in Eq. (12).

In [18] it has been shown that an additional dipole term improves the intrinsic cutoff scale, moreover it has been shown in [17] [19] that, if higher powers of the background gauge field are dropped, an additional dipole term has to be included. Such dipole term in the Abelian case reads (we follow the notation of [18])

$$L_{dipole} = ie\alpha F_\nu^\mu H_{\mu\rho}^* H^{\rho\nu} + c.c. , \quad (16)$$

$$H_{\mu\rho}^* = h_{\mu\rho}^* + \frac{1}{m} (D_\mu B_\rho^* + D_\rho B_\mu^*) - \frac{1}{2m^2} (D_\rho D_\mu + D_\mu D_\rho) \phi^* . \quad (17)$$

The non-Abelian dipole term is the obvious generalization of the above one:

$$L_{dipole} = \alpha g f^{abc} F_\nu^{a\mu} H_{\mu\rho}^b H^{c\rho\nu} \quad (18)$$

Although such a term does not affect the analysis of the Gribov problem, it does affect the computations of physical quantities such as cross sections. In the following we will give an example of such computations both with and without the dipole term.

At this point it is convenient to specialize our discussion by assuming that the gauge group of our theory is the usual color group $SU(3)_c$ and A_μ is nothing else but the gluon. Starting from Lagrangian (12), it is, then, possible to obtain the Feynman rules needed for computing the double production of the color-octet spin-2 particles at the LHC, at tree level. We focus on double production since, as in any gauge theory, all the interaction terms contain the matter field (in this case the spin-2 field) in pairs. In principle, interaction terms coupling, for instance, two gluons to a single spin-2 particle are possible. Nevertheless, we recall, such an interaction is not originated by the gauge principle and introduces an UV and model independent cut-off [18] and theoretical uncertainties. In this sense, the situation is similar (although more severe) to the single production of spin-1 color-octet vector resonance which is plagued of theoretical uncertainties [52].

So, we used the package FeynRules [53] to obtain the relevant Feynman rules of the model and Madgraph 5 [?] in order to compute the cross section. The Feynman diagrams are shown in figure 1

First, we computed the cross sections, without taken into account the dipole term, for different masses of the spin-2 particle in the range [500, 2000] GeV. The center-of-mass energy was assume to

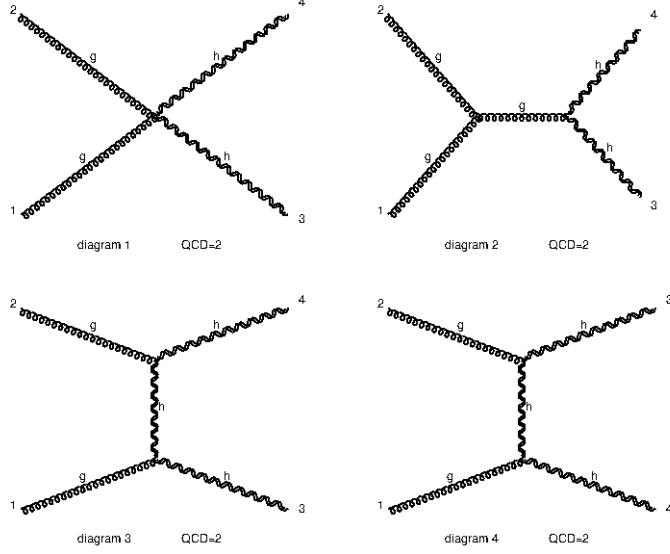


Figure 1: Feynman diagrams corresponding to the double production of the color-octet spin-2 particles (h). Figure produced by Madgraph

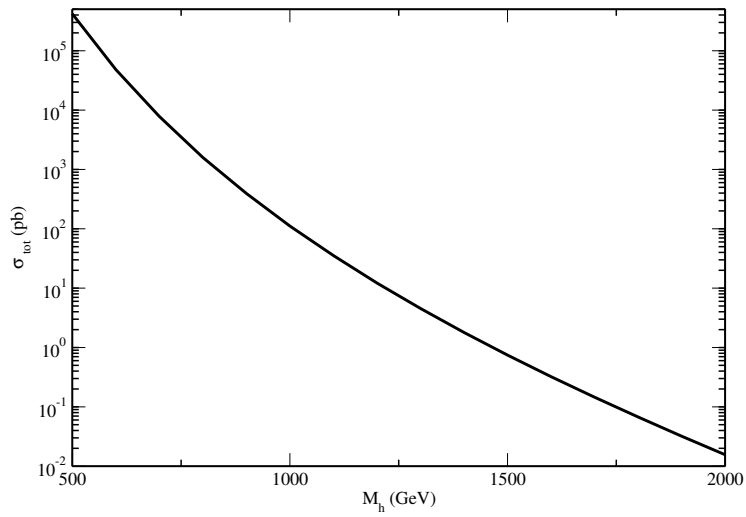


Figure 2: Cross sections for the double production of color-octet spin-2 with mass (M_h) in the range $[500, 2000]$, at the LHC. We assume $\sqrt{s} = 14$ TeV and a QCD coupling constant given by $g_s = 1.2$

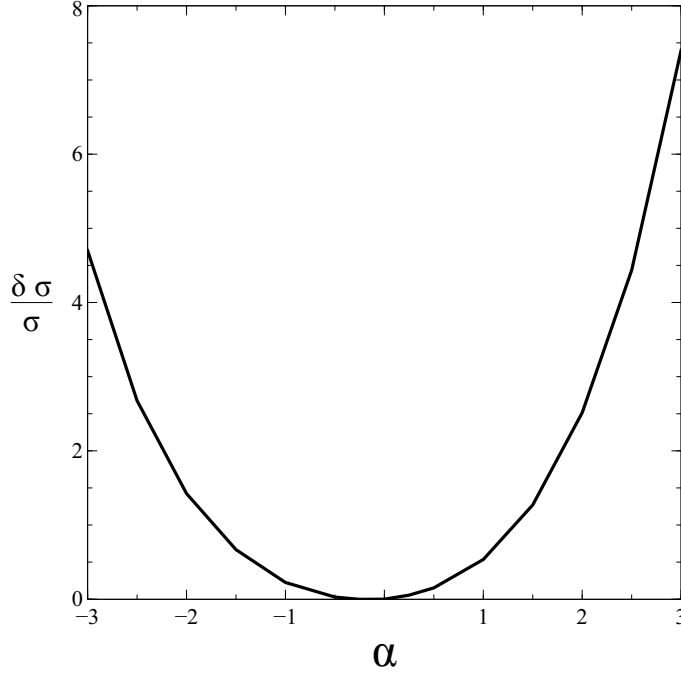


Figure 3: Effect of the dipole term on the production cross section of two spin-2 particles at the LHC for $M_h = 1$ TeV

be $\sqrt{s} = 14$ TeV and we used the CTEQ6L parton distribution function. The obtained cross sections are shown in figure 2.

Our numerical results are in agreement with the ones reported in [1] which were obtained by a completely different method.

However the inclusion of the dipole term, which is well motivated from the theoretical point of view, may significantly modify these results. In order to evaluate this effect, we computed the cross sections in the context of the LHC as above for different values of the α parameter ($\sigma(\alpha)$) but taking a fixed value for the mass of the spin-2 particle ($M_h = 1$ TeV). In figure 2, we show the effect of the dipole term by plotting the quantity

$$\frac{\delta\sigma}{\sigma} = \frac{\sigma(\alpha) - \sigma(\alpha = 0)}{\sigma(\alpha = 0)} \quad (19)$$

where $\sigma(\alpha)$ is the cross section computed with the dipole term and $\sigma(\alpha = 0)$ is the cross section without the dipole.

Notice that for the preferred value of α , at least in the Abelian case, ($\alpha = -1/4$) the departure from the $\alpha = 0$ case is negligible.

In order to proceed, the Stückelberg symmetry has to be gauge-fixed.

3 Review of the SU(N) Gribov problem and its IR nature

The degrees of freedom of any gauge theory are encoded in a Lie algebra valued one form $(A_\mu)^a$. The action functional is invariant under finite gauge transformations

$$A_\mu \rightarrow U^{-1} A_\mu U + U^{-1} \partial_\mu U \quad (20)$$

whereas the physical observables are invariant under proper gauge transformations. Unfortunately (besides the cases of topological field theories in 2+1 dimensions [26]), it is still unknown how to use in practice gauge invariant variable in Yang-Mills case. Hence, the usual recipe to fix the gauge and to use perturbative expansion around the trivial vacuum $A_\mu = 0$ provides one with excellent results when the coupling constant is small. The most convenient practical choices for the gauge fixing are the Coulomb gauge and the Landau gauge³. However Gribov showed [30] that a *proper gauge fixing* is not possible globally and that, in the QCD case, this global effect is very important in the non-perturbative regime. Furthermore, it has been shown by Singer [32], that if Gribov ambiguities occur in Coulomb gauge, they occur in all the gauge fixing conditions involving derivatives of the gauge field. In the path integral formalism, one has to pay close attention to the issue of Gribov copies. Indeed, as it is well known, there is a close relation between *gauge fixing ambiguities* and *smooth zero modes of the Faddeev-Popov (FP) operator*. Furthermore, even if one chooses a gauge free of Gribov copies, the effects of Gribov ambiguities in other gauges generate a breaking of the BRST symmetry [41].

One can also verify that all the *simple-minded hopes* that the Gribov problem can be solved automatically by the path-integral formalism itself fail. For instance, naively, one could think that if one performs the path integral without any restriction, the contributions coming from the non-trivial copies cancel against each others and one would be left with a sum in which there is just one term for each gauge orbit. Unfortunately, the reality is totally different and one is confronted with the so-called Neuberger 0/0 problem [29]. Namely, in the most direct translation of BRST symmetry on the lattice, there is a perfect cancellation among these gauge copies. Consequently, the expectation value of any gauge invariant (and thus physical) observable in a lattice BRST formulation will always be of the indefinite form 0/0 and so ill-defined and, as it is well known, the formulation of the continuous theory does not help either.

The arising of Gribov copies can be described as a bifurcation problem. Let A_μ a gauge potential in the Landau gauge and $(A^U)_\mu$ a potential gauge-equivalent to A_μ . In the case of a perfect gauge fixing it should happen that the system of equations below

$$\partial_\mu A^\mu = 0 \ , \quad (21)$$

$$\partial_\mu (A^U)^\mu = \partial_\mu (U^{-1} A^\mu U + U^{-1} \partial^\mu U) = 0 \ , \quad (22)$$

has a unique trivial solution $U = \mathbf{1}$ for any A_μ satisfying Eq. (21). In other words, one should hope that there is no smooth globally defined gauge transformation U satisfying Eq. (22). As Gribov showed [30] this is not true. There are known results in the theory of bifurcation (in particular, the so-called Krasnosel'skii's theorem [64]) which provide one with sufficient conditions for the appearance of Gribov copies. As it will be now shown, such conditions have a nice physical interpretation. In rough terms, the Krasnosel'skii's theorem can be stated as follows: write the U in Eq. (22) as Taylor expansion

$$U = \mathbf{1} + \alpha + R(\alpha) \quad (23)$$

where $\mathbf{1}$ is the identity of the gauge group and $R(\alpha)$ contains terms of order α^2 or higher. Replacing the above expansion in Eq. (22) one gets

$$\partial_\mu (A^U)^\mu = (\partial_\mu D^\mu) \alpha + T(\alpha) = 0 \ ,$$

where the operator $T(\alpha)$ (which encodes the non-linear part of Eq. (22)) has the property that

$$T(\alpha) \xrightarrow{\alpha \rightarrow 0} 0 \ .$$

³The axial and light-cone gauge fixings are affected by some non-trivial technical problems in implementing the physical boundary conditions on the gauge fields: see, for instance, [28]. Moreover, it is unclear how to carry on the " $-i\epsilon$ " prescription in the propagators beyond one-loop computations (see [33] for a detailed review).

Under some technical assumptions (which are usually verified in situations which are physically relevant) the Krasnosel'skii's theorem tells that, in order to understand whether or not non-trivial solutions of Eq. (22) appear, it is enough to look at the linear part of the equation. In particular, if the equation

$$\begin{aligned} (\partial_\mu D^\mu) \alpha &= 0 , \\ D_\mu &= D(A)_\mu = \partial_\mu + [A_\mu, \cdot] , \end{aligned} \quad (24)$$

has a smooth normalizable solution then a non-trivial solution of Eq. (22) will appear. The smooth normalizable solutions of Eq. (24) can be called *small Gribov copies*. It is well known that Eq. (24) (which is nothing but the equation for the zero-modes of the Faddeev-Popov operator) will have non-trivial solution if the gauge potential is “large enough”⁴ with respect to the a suitable L_2 norm [38]:

$$\|A_\mu\| = \int d^4x \text{Tr} A^2 . \quad (25)$$

Such norm induces in a natural way the following norm for the zero modes α of the Faddeev-Popov operator:

$$\|\alpha\| = \int d^4x \text{Tr} (D_\mu \alpha)^2 \quad (26)$$

The above considerations clarify why, in the case of $SU(N)$ gauge theories the Gribov is “Infra-Red” in nature. The reason is that the relevant quantity for the appearance of zero modes is the above norm for A_μ . In order for the norm to be large enough, it is not necessary that, when one expands A_μ in Fourier series, there are many Fourier modes with high (Euclidean) four-momentum k_μ . In other words, A_μ can have a large enough norm even if it is a very slowly varying function with no Fourier mode with large k_μ . Hence, the Gribov problem in $SU(N)$ gauge theories is an IR issue (of course, this is not the case in gravity but we will focus on the Yang-Mills case here). On the other hand, the requirement to have finite norm in the above sense has been often criticized (see, for an up-to-date discussion [66]). In particular, it is possible to construct gauge fields which have finite energy and/or action (and, therefore, which should not be discarded) but infinite norm in Eq. (25). This suggests that to impose the finiteness of the norms in Eqs. (25) and (26) could be a too severe restriction. In particular, the most conservative requirements that can be imposed to avoid infinitesimal Gribov copies is to ask that both the norm and the energy of the gauge potential must be finite. We will come back on this point in the next section.

A very elegant solution of the Gribov problem (see, in particular, [30] [34] [35] [36] [37] [38]; two nice reviews are [39] [40]) has been the restriction of the path integral to the region Ω around $A_\mu = 0$ in which the FP operator is positive (called Gribov region)

$$\Omega \stackrel{def}{=} \left\{ A_\mu \mid \partial^\mu A_\mu = 0 \text{ and } \det \partial^\mu D(A)_\mu > 0 \right\} . \quad (27)$$

In the case in which the space-time metric is flat and the topology is trivial this approach coincides with usual perturbation theory when the gauge field A_μ is close to the origin. At the same time, this framework takes into account the Infra-Red effects related to the partial [38] elimination of the Gribov copies [35] [42] [43]. When one introduces suitable condensates [44] [45] [46] [47] [50] the agreement with lattice data is excellent [48] [49]. Interestingly enough, within this framework, one can also solve the old problem of the Casimir energy in the MIT-bag model [59]. The semiclassical approach *a la Gribov* works very well at finite temperature as well providing one with a good description of the phase diagram and of the deconfinement transition with results in good agreement with lattice data [60] [61] [62].

⁴In this case, “large enough” means large enough compared to the first eigenvalue of the four-dimensional Laplacian.

On flat space-times with trivial topology, this possibility is consistent with the usual perturbative point of view since, in the case of $SU(N)$ gauge theories, it has been shown that there exist *a neighborhood of $A_\mu = 0$ in the functional space of the gauge potential* (with respect to a suitable functional norm [38]) which is free of Gribov copies in the Landau or Coulomb gauge. On the other hand, the pattern of appearance of Gribov copies strongly depends on the space-time metric and topology and the situation can become much more complicated. (see, in particular, [51] [55] [56] [57] [58] [63]).

4 The gauge fixing problems of the Porrati-Rahman action

The unique gauge fixing choice [18] for the Stückelberg symmetry which allows to diagonalize properly the kinetic terms (in such a way to provide a unified description of phenomena such as strong coupling at finite energy scale, acausal propagation in external fields and so on) is

$$F_\mu^a = \partial^\nu h_{\mu\nu}^a - \frac{1}{2} \partial_\mu h^a + m B_\mu^a = 0 , \quad (28)$$

$$F^a = \partial^\nu B_\nu^a + \frac{m}{2} (h^a - 3\phi^a) = 0 . \quad (29)$$

As it is easy to see, locally this is a good gauge fixing for the Stückelberg symmetry since, *at a first glance*, neither F_μ^a nor F^a are invariant under the Stückelberg gauge transformations in Eqs. (7), (8) and (9). Indeed, under the Stückelberg gauge transformations the gauge-fixing conditions in Eqs. (28) and (29) change as

$$\delta F^a = ((\partial_\mu D^\mu - 3m^2) \lambda)^a + m ((D^\mu - \partial^\mu) \lambda_\mu)^a , \quad (30)$$

$$\delta F_\mu^a = \partial^\nu (D_\mu \lambda_\nu + D_\nu \lambda_\mu)^a - \partial_\mu (D_\nu \lambda^\nu)^a + m (D_\mu \lambda - m \lambda_\mu)^a , \quad (31)$$

and for *generic* λ and λ_μ one has that

$$\delta F^a \neq 0 , \quad \delta F_\mu^a \neq 0 .$$

Hence, for obvious reasons, we will call *infinitesimal Gribov copies* (λ_μ, λ) *corresponding to the Stückelberg gauge transformations* the non-trivial solutions of the system $\delta F^a = 0$ and $\delta F_\mu^a = 0$. The system of equations for the appearance of infinitesimal Gribov copies explicitly reads

$$((\square_{FP} - 3m^2) \lambda)^a + m ((D^\mu - \partial^\mu) \lambda_\mu)^a = 0 , \quad (32)$$

$$(\square_{FP} \lambda_\mu)^a - m^2 \lambda_\mu^a + \partial^\nu (D_\mu \lambda_\nu)^a - \partial^\mu (D_\nu \lambda^\nu)^a + m (D_\mu \lambda)^a = 0 , \quad (33)$$

$$\square_{FP} = \partial_\mu D^\mu , \quad \square = \partial_\mu \partial^\mu ,$$

where in the following the metric will be assumed to be flat and Euclidean. Obviously, the existence of non-trivial solutions of the above system which are *smooth everywhere and satisfy reasonable boundary conditions* implies that the gauge-fixing procedure is not well-defined. As it has been already emphasized, unlike the pathology analyzed in [18], this Gribov-like ambiguity appears in the IR. Here, we will analyze only the Abelian case since it already contains all the physical ingredients (however, the present results can be easily generalized to the non-Abelian case).

The worst gauge fixing pathology which in principle can arise corresponds to the situation in which non-trivial solutions appear even when $A_\mu = 0$ in Eqs. (32) and (33). Indeed, if Eqs. (32) and (33) would have smooth normalizable solutions for vanishing potential background, then this would

invalidate any perturbative approach to analyze such theory. Fortunately in this case it can be shown that when $A_\mu = 0$ no smooth normalizable solutions of Eqs. (32) and (33) exist. In particular, when $A_\mu = 0$, Eq. (32) becomes

$$(\square - 3m^2) \lambda = 0 , \quad (34)$$

which does not admit non-trivial solutions due to the fact that the eigenvalues of Laplacian operator (with any reasonable boundary conditions⁵) has positive eigenvalues. Therefore Eq. (34) implies that $\lambda = 0$. It is worth to point out that also if we use the lagrangian of [25] instead of the one of [18] it will not change the further discussion as we will from now on focus on the sector with $\lambda = 0$. If one replaces $\lambda = 0$ into Eq. (33) then one gets

$$(\square - m^2) \lambda_\mu = 0 ,$$

and, due to the positivity of the spectrum of the Laplacian, the only solution is $\lambda_\mu = 0$. Hence, these simple considerations show that in order to have gauge fixing ambiguity for the Stückelberg symmetry A_μ must deviate enough from 0 in close analogy with the Gribov-Zwanziger scenario.

It is worth to emphasize the following important point. In the discussion of Gribov copies in QCD *there are only two key players*: the dynamical field (that is, the gauge potential A_μ) and the gauge parameter α (see Eqs. (24) and (25)). Indeed, the appearance of gauge fixing ambiguities of the Yang-Mills gauge symmetry only depends on the norm in Eq. (25) characterizing the Yang-Mills gauge potential itself. In the case of the gauge fixing for the Stückelberg symmetry, there are *three key players*: the dynamical fields ($h_{\mu\nu}$, B_μ and ϕ), the gauge parameters of the Stückelberg symmetry (that is, λ and λ_μ) *and* A_μ . This is a huge difference: the appearance of gauge fixing ambiguity in the Stückelberg case (namely, smooth normalizable solutions of Eqs. (32) and (33)) *does not depend on suitable norms of* $h_{\mu\nu}$, B_μ and/or ϕ (as one would expect on the basis of a naive analogy with the standard case). In fact, as it will be now shown, the appearance of gauge fixing ambiguity in the Stückelberg case *does depend on the norm of* A_μ which is the third key player in the analysis of the gauge fixing of the Stückelberg symmetry. In other words, whether or not gauge fixing ambiguities appear in the Stückelberg case *depends on the (norm of a) gauge potential of another gauge symmetry* (Maxwell in the present case). This shows that the analysis of the Gribov phenomenon in the Stückelberg case is more complicated than in the usual cases due to the fact that it depends heavily with the interactions with another (non-Stückelberg) gauge field. In the appendix we will also show that a very similar Gribov-like ambiguity appears in the case of Proca fields.

For simplicity, one can assume λ_μ as orthogonal to A_μ in such a way that the above system reduces to

$$(\square_{FP} - 3m^2) \lambda = 0 , \quad (35)$$

$$\square_{FP} \lambda_\mu - m^2 \lambda_\mu + \partial^\nu (D_\mu \lambda_\nu) - \partial^\mu (D_\nu \lambda^\nu) + m D_\mu \lambda = 0 , \quad (36)$$

$$A_\mu \lambda^\mu = 0 , \quad (37)$$

where, as it has been already explained in the previous sections, the gauge parameter λ and λ_μ are charged under the Maxwell gauge symmetry. One can take $\lambda = 0$ obtaining the following system of equations

$$\square_{FP} \lambda_\mu - m^2 \lambda_\mu + \partial^\nu (D_\mu \lambda_\nu) - \partial^\mu (D_\nu \lambda^\nu) = 0 . \quad (38)$$

in which the gauge field plays the role of an external background field.

⁵In mathematical textbooks, the eigenvalue equation for the Laplacian is usually written as: $(\Delta + \lambda) u = 0$, where $\Delta = \sum_i \partial_i^2$. With this convention, all the eigenvalues λ_i are positive: $0 < \lambda_1 \leq \lambda_2 \leq \dots$. Consequently, the operator $(\square - 3m^2)$ in Eq. (34) is invertible and the homogeneous equation has only the trivial solution.

We will now show that gauge fixing ambiguities may arise when the gauge potential background A_μ is “large enough”. In particular, we will focus on the most interesting case in which A_μ is locally a pure gauge. The interest of this choice is that a pure gauge will pass even the most severe requirements which are usually imposed to avoid the Gribov issue (see the discussion in the previous section). In particular, a pure gauge has finite energy and action. Hence, once it is found that a pure gauge potential can support non-trivial solutions of Eq. (38), there is no reasonable physical boundary condition which can justify the omission of such solution. Obviously, by enlarging the class of possible ansätze for A_μ one would also enlarge the number of different non-trivial solutions of Eq. (38) but our choice is the one which discloses in the clearest possible way the origin of the phenomenon.

In order to find such copies it is useful to notice that in the case of [18], Eq. (38) reads

$$\partial_\nu (ieA^\nu \lambda_\mu) + \partial^\nu (ieA_\mu \lambda_\nu) = -\square \lambda_\mu + m^2 \lambda_\mu . \quad (39)$$

Let us analyze the $U(1)$ Stückelberg theory within a four-dimensional torus T^4 (this case corresponds to put the system at finite temperature and in a finite space volume). Such a situation is not just of academic interest since these circumstances are achieved, for example, in relativistic heavy ion collisions as those experimentally studied at RHIC and the LHC (Pb-Pb mode). The natural boundary conditions for λ_μ are the periodic ones. Let us consider in Eq. (39) a pure gauge potential $A_\mu = \text{const}$ (which obviously satisfies periodic boundary conditions). It is enough to consider λ_μ where only two components are switched on, for example λ_1 and λ_2 where both functions depend only on the variables (x_1, x_2) . Using as ansatz

$$\lambda_1 = e^{\alpha x_1} e^{\beta x_2} \quad (40)$$

The orthogonality condition $A_\mu \lambda^\mu = 0$ implies

$$\lambda_2 = -\frac{A_1}{A_2} \lambda_1 = -\frac{A_1}{A_2} e^{\alpha x_1} e^{\beta x_2} \quad (41)$$

Choosing λ_μ to be divergence free i.e. $\partial_\mu \lambda^\mu = 0$ we get

$$\beta = \frac{A_2}{A_1} \alpha \quad (42)$$

so that eventually we get

$$\lambda_1 = e^{\alpha x_1} e^{\frac{A_2}{A_1} \alpha x_2} \quad ; \quad \lambda_2 = -\frac{A_1}{A_2} e^{\alpha x_1} e^{\frac{A_2}{A_1} \alpha x_2} . \quad (43)$$

Taking into account that A_μ must be real, it is then straightforward to check that (40) and (41) are solutions of (39) which satisfy the correct boundary conditions (i.e. α and β are purely imaginary) only if

$$e^2 (A_1^2 + A_2^2) = e^2 A_\mu A^\mu > 4m^2 . \quad (44)$$

Hence, smooth solutions of Eq. (39) satisfying the periodic boundary conditions (which represent infinitesimal Gribov copies of the Stückelberg gauge symmetry within T^4) appear when the gauge potential is large enough.

Interestingly enough, the condition in Eq. (44) is very similar to the usual condition determining the appearance of Gribov copies in the Yang-Mills case (when the product “coupling constant time gauge potential” must be large enough to give rise to infinitesimal copies). The present Stückelberg Gribov copies are related *both* to the presence of the massive spin two field *and* to the fact that the

gauge parameter is charged under another (non-Stückelberg) gauge group.

Since (at least in the rather extreme case we have considered) only “large” constant gauge potentials generate infinitesimal Gribov copies of the Stückelberg gauge symmetry an intriguing possibility arises. One could solve the problem using also in this case a Gribov-Zwanziger like restriction. However, as it has been already explained, the implementation of the Gribov-Zwanziger restriction in the present case is more complicated than in the QCD case. On the other hand, it is interesting to note that one obvious consequence of such restriction would be the appearance of non-local propagators for the higher spin massive particles. In this sense, this would not be a surprise since, within the approach developed in [21], massive higher spin particles can be described without auxiliary fields the price to pay being the appearance of non-local propagators. We hope to analyze in future the relations between the two approaches.

5 Conclusions and further comments

In this paper we analyzed the interactions of a massive spin-2 particles charged under both Abelian and non-Abelian group using the Porrati-Rahman Lagrangian. Moreover, a scalar field is needed in 4D due to the presence of an ambiguity in minimal coupling. It has been shown in [17] [19] that if higher powers of the background gauge field are dropped, an additional dipole term has to be included. In this way, the well-known inconsistencies (like acausality and wrong number of propagating degrees of freedom) are under control provided one uses the action only as an effective action with a characteristic UV cutoff scale. We have shown that, besides the well understood UV cut-off (signaling the arising of pathologies such as the Velo-Zwanziger problem), this Lagrangian has also some non-trivial IR issues. Their origin is a gauge-fixing ambiguity akin of Gribov copies in QCD for the Stückelberg symmetry in an Abelian background. This type of ambiguity prevents a global covariant gauge fixing of the Stückelberg symmetry (which is the only gauge-fixing choice unifying phenomena such as strong coupling at finite energy scale, acausal propagation in external fields and so on). Explicit examples have been found when the theory is analyzed within a finite volume. In this case, we have constructed Gribov copies corresponding to the Stückelberg gauge symmetry supported by Abelian gauge potentials with zero field strength (but large enough norm). To the best of authors knowledge, this is the first analysis of the peculiar features of the Gribov problem for the Stückelberg gauge symmetry. At least in the case of a constant gauge potential, gauge fixing ambiguities only arise when the gauge potential is “large enough” (as it happens in QCD). From the phenomenological point of view, this is a fortunate circumstance since it allows perturbative analysis, like the one presented in section 2, which can be useful for the experimental search of colored spin-2 states predicted by some models. From a theoretical perspective, it is natural to wonder whether the Gribov-Zwanziger strategy of restricting the path integral to a copy free region can be applied in this case as well. We hope to come back on this interesting issue in the future.

6 Acknowledgments

The authors thank Dario Francia and the anonymous referee for their many useful comments and suggestions which improved a lot this manuscript. This work is partially supported by FONDECYT grants 1160137, 1150246 and 1160423, Conicyt Proyecto Basal FB0821 and ACT1406. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence

Base Financing Program of CONICYT.

A The Proca field

In this appendix we will show that a similar Gribov-like phenomenon also appears for Proca fields. The results in the main text together with the discussion in this appendix suggest that this phenomenon could be relevant for higher spin fields as well. Here we will follow the presentation of the charged Proca field outlined in [18]. Let us consider the usual complex Proca Lagrangian

$$L = -\frac{1}{2}G_{\mu\nu}^*G^{\mu\nu} - m^2W_\mu^*W^\mu, \quad G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu.$$

The action becomes gauge invariant after the replacement

$$W_\mu \rightarrow V_\mu - \frac{\partial_\mu \phi}{M}.$$

Then, as in the spin-2 case discussed in the main text, the coupling with a $U(1)$ field A_μ is achieved introducing the covariant derivatives

$$\partial_\mu \rightarrow D_\mu = \partial_\mu \pm ieA_\mu.$$

One can obtain a diagonal kinetic term including the gauge-fixing term

$$L_{gf} = -|\partial_\mu V^\mu - M\phi|^2. \quad (45)$$

Once again, it is worth emphasizing that unlike what happens in Yang-Mills theory, in the present case such a gauge-fixing is mandatory in order to have a well-defined kinetic term.

However, due to the coupling with the A_μ field, the gauge-fixing in Eq. (45) does have Gribov copies. Following the same steps as in Eqs. (28), (29), (30), (31), (32) and (33) one gets the following *equation for the Gribov copies of the charged Proca field*:

$$\partial_\mu D^\mu \alpha - M^2 \alpha = 0, \quad (46)$$

where α is the complex $U(1)$ gauge parameter. As in the case analyzed in the main text (see Eq. (34)), non-trivial solutions only appear when the $U(1)$ gauge field is "large enough" since Eq. (46) has only trivial solutions when $A_\mu = 0$ (and, by continuity, when it is small). On the other hand, it is easy to construct non-trivial solutions following the same approach outlined in the previous sections in T^4 . Moreover, in the present case of charged Proca fields, it is also possible to construct many explicit examples of copies by applying the Henyey strategy [67] (which cannot be applied so easily in the spin-2 case). Namely, one can interpret Eq. (46) as an equation for A_μ choosing, a priori, the α . In this way, one obtains explicit examples of copies together with the corresponding $U(1)$ gauge fields supporting them. Indeed, in the case of the Proca field, it is simpler than in the spin 2 case to find explicit copies as using the Henyey approach one has only two equations to solve—one for the real and one for the imaginary parts of α —and four components of A_μ at one's disposal to play with.

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